Exercise 47

Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

Solution

The defining equation for a circle with radius R that's centered at O is

$$x^2 + y^2 = R^2$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(R^2)$$

Use the chain rule to differentiate $y = y(x)$.

$$2x + 2y\frac{dy}{dx} = 0$$

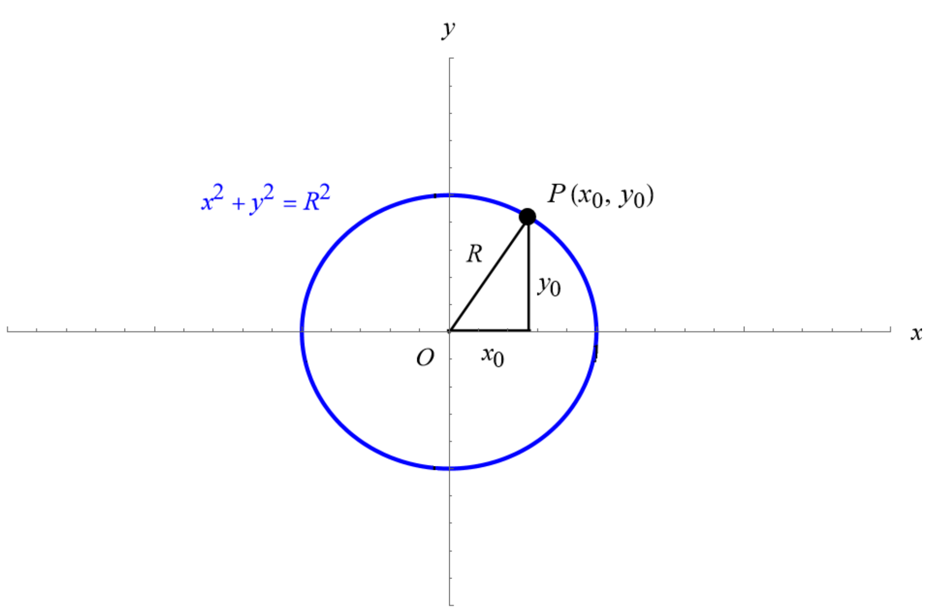
Solve for dy/dx.

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The slope of the tangent line at the point (x_0, y_0) is then

$$m = -\frac{x_0}{y_0}.$$



Observe that the slope of line OP is

$$m_{OP} = \frac{y_0}{x_0},$$

the rise over run. These two slopes are negative reciprocals. Therefore, any tangent line at a point P to a circle with center O is perpendicular to the radius OP.